FUNDAMENTAL GROUP OF SELF-DUAL
FOUR-MANIFOLDS WITH POSITIVE SCALAR CURVATURE

BY

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ABSTRACT

In the first part of the paper we show in a very simple manner that the fundamental groups of four-manifolds with a self-dual metric of positive scalar curvature admit a surjective homomorphism on $\mathbb{Z}$, if there is a nontrivial representation. In the second part we outline a method how to construct negatively curved metrics in Taubs collars.

1. Background

It is a fundamental goal in Riemannian geometry to understand the topology of manifolds of positive curvature. The only general facts so far known are: finiteness of the fundamental group (Myers' theorem), vanishing of the $\hat{A}$-genus (Lichnerowicz' theorem and the modification of Hitchin) and a universal bound on Betti numbers (Gromov's theorem). In a well-known paper [13] Micallef and Moore introduced a new notion of positivity for the curvature tensor, that is, *positivity on complex isotropic two-planes*. For $x \in M$, a Riemannian manifold, let $R: \Lambda^2 T_x M \to \Lambda^2 T_x M$ be the curvature tensor. After complexification we get a Hermitian operator in $\Lambda^2 T_x^{\mathbb{C}} M$. We say that $z \in \Lambda^2 T_x^{\mathbb{C}} M$ comes from a complex isotropic two-plane if $z = \xi \wedge \eta$ with $(\xi, \xi)_{\mathbb{C}} \doteq (\xi, \eta)_{\mathbb{C}} = (\eta, \eta)_{\mathbb{C}} = 0$. Here $(\cdot, \cdot)_{\mathbb{C}}$ is the canonical symmetric (not Hermitian!) complexification of the Euclidean scalar product in $T_x M$. The condition above says that $(Rz, z) > 0$ for such z. The theorem of Micallef and Moore reads:

1.1. THEOREM (Micallef, Moore): *Let M^n be a compact, simply connected Riemannian manifold. If $(Rz, z) > 0$ for z as above, then M is homeomorphic to S^n.*

A remarkable application is a pointwise pinching theorem, which reads as follows:

1.2. COROLLARY: *Let M be a compact Riemannian manifold, whose sectional curvature satisfies $\frac{1}{4}B(x) < K(x) \leq B(x)$ for some positive function B; then $M \approx S^n$.*

If one allows equality in $(Rz, z) \geq 0$ or $\frac{1}{4}B(x) \leq K(x) \leq B(x)$, one encounters more topological types of manifolds, like complex projective spaces. In fact, Cao [4] classified compact simply-connected four-manifolds with curvature, nonnegative on complex isotropic two-planes. On the other hand LeBrun [3] classified the underlying topological manifolds of simply-connected four-manifolds of positive scalar curvature.

In both cases, given the absence of the Myers theorem, cited above, one asks an important question:

1.3 WHICH FUNDAMENTAL GROUP MAY MANIFOLDS WITH SUCH A CURVATURE HAVE? Below in Theorem 3.3 we will derive a very strong restriction on $\pi_1(M)$ in case of dimension four, namely:

MAIN THEOREM (3.3): *Let M be a compact four-dimensional manifold either with curvature, positive on complex isotropic two-planes, or self-dual of positive scalar curvature. If $\pi_1(M)$ admits a nontrivial unitary representation, and M is orientable, then there exists a surjective homomorphism from $\pi_1(M)$ on $\mathbb{Z}$.*

COROLLARY: *If $\pi_1(M)$ is finite, then either $\pi_1(M) = 1$ or $\pi_1(M) = \mathbb{Z}_2$.*

Observe that finitely presented groups which do not admit a nontrivial unitary representation, are extremely rare (see 3.4).

We will also discuss the diffeomorphism problem, namely

1.4. IS IT TRUE, THAT UNDER THE CONDITIONS OF THEOREM 1.1, M IS DIFFEOMORPHIC TO S^n? In connection with this question, recall that some exotic spheres admit a metric of positive sectional curvature [10]. On the other hand, "strongly" pinched manifolds are standard spheres [12], [14]. Here, we

suggest a completely new approach to this problem, using the theory of SD-connections on four-manifolds [8], especially the Donaldson's collar theorem.

2. Positivity of curvature on complex isotropic two-planes: an algebraic study

2.1 The computation that follows is essentially contained in the Micallef and Moore paper. Let V be a four-dimensional orientable Euclidean space. Consider the canonical decomposition $\Lambda^2 V = \Lambda_+^2 V \oplus \Lambda_-^2 V$. The unit sphere $S_\pm^2(V)$ in $\Lambda_\pm^2(V)$ consists of twistors, that is, orthogonal complex structures in V. Let $Z \subset V^{\mathbb{C}}$ be a complex isotropic two-plane. Since $Z \cap V = Z \cap iV = 0$, we may look at Z as a graph of an invertible real operator $\mathcal{P}\colon V \to V$. The equation $(v + i\mathcal{P}v, w + i\mathcal{P}w) = 0$ for any $v, w \in V$ implies $\mathcal{P} \in S_\pm^2(V)$. Next, let $R\colon \Lambda^2 V \to \Lambda^2 V$ be a curvature-like symmetric operator. We denote the Hermitian complexification of R again by R. Choose unit vectors v, w in V such that $(v, w) = (\mathcal{P}v, w) = 0$. The complex plane, corresponding to $\mathcal{P}$, is spanned by $v + i\mathcal{P}v$ and $w + i\mathcal{P}w$, so the element z in $\Lambda_{\mathbb{C}}^2 V$ is $(v \wedge w - \mathcal{P}v \wedge \mathcal{P}w) + i(\mathcal{P}v \wedge w + v \wedge \mathcal{P}w)$. Assume $\mathcal{P} \in S_+^2(V)$. Then the bivectors

$$f = \frac{1}{\sqrt{2}}(v \wedge w - \mathcal{P}v \wedge \mathcal{P}w) \quad \text{and} \quad g = \frac{1}{\sqrt{2}}(\mathcal{P}v \wedge w + v \wedge \mathcal{P}w)$$

are complementing elements of an orthonormal basis $(f, g, \mathcal{P})$ of $\Lambda_+^2(V)$. The condition $(Rz, z) \geq 0$ therefore reads $(Rf, f) + (Rg, g) \geq 0$, or $\operatorname{Tr} R\lceil_{\Lambda_+^2(V)} \geq R(\mathcal{P}, \mathcal{P})$. We will state this in the form of a lemma, in which $s = 4 \operatorname{Tr} R|_{\Lambda_\pm^2(V)}$ is scalar curvature, and W is the Weyl tensor.

2.2 LEMMA: *A curvature-like tensor $R\colon \Lambda^2 V \to \Lambda^2 V$ is nonnegative on complex isotropic two-planes, if and only if for a unit vector $\mathcal{P}$ of $\Lambda_\pm^2(V)$ one has $(W\mathcal{P}, \mathcal{P}) \leq \frac{s}{6}$.*

3. Vanishing results and the fundamental group

3.1 Consider the modified de Rham complex [8],

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{\mathcal{D}} \Omega_\pm^2(M),$$

where $\Omega^i(M)$ stands for the sheaf of $C^\infty i$-forms on M, and $\Omega_\pm^2(M)$ is a sheaf of sections of $\Lambda_\pm^2 T^*(M)$.

The Weitzenböck formula [1], [8] gives

$$\mathcal{D}^*\mathcal{D} = \nabla^*\nabla - 2W^\pm + \frac{s}{3}.$$

Accounting 2.2, we conclude (see also [4]):

PROPOSITION (3.1): *Let M be a compact four-manifold with a curvature, non-negative on complex isotropic two-planes, or a self-dual manifold with non-negative scalar curvature. Then any self-dual or anti-self-dual harmonic two-form on M (resp. anti-self-dual form) is self-parallel. If the curvature above is positive, then $H^2(M, \mathbb{R}) = 0$. (resp. $H^2_-(M, \mathbb{R}) = 0$).*

3.2   Here we will derive twisted versions of 3.1. Let $\rho\colon \pi_1(M) \to U(n)$ be a unitary representation. Let E_ρ be the corresponding flat Hermitian vector bundle over M. Let $(\Omega^*(M, E), d)$ be the complex of E-valued forms, where d is induced by the flat connection. The cohomology of $\Omega^*(M, E)$ coincides with the cohomology of the local system associated to ρ. One has the Laplace operator Δ_i acting in $\Omega^i(M, E)$ and the Hodge theorem $\mathrm{Ker}\,\Delta_i \approx H^i(M, E)$. The proof of the following theorem is parallel to the proof of 3.1.

THEOREM (3.2): *Let M be as in 3.1. Then any self-dual or anti-self-dual harmonic two-form with coefficients in E (resp. anti-self-dual harmonic two-form with coefficients in E) is self-parallel. If the curvature on complex isotropic two-planes is positive (resp. the scalar curvature is positive), then $H^2(M, E) = 0$ (resp. $H^2_-(M, E) = 0$).*

3.3   Now, assume that M is a compact four-manifold with curvature positive on complex isotropic two-planes or a self-dual manifold with positive scalar curvature. Then by Theorem 3.1, $H^2(M, \mathbb{R}) = 0$ (resp. $H^2_-(M, \mathbb{R}) = 0$) and by Theorem 3.2, $H^2(M, E_\rho) = 0$ (resp. $H^2_-(M, E_\rho) = 0$) for any unitary representation ρ. This implies the following strong statement concerning the structure of $\pi_1(M)$.

MAIN THEOREM (3.3): *Suppose M is orientable and there exists a nontrivial unitary representation of $\pi_1(M)$. Then there exists a surjective homomorphism $\pi_1(M) \to \mathbb{Z}$.*

Proof: Suppose $H^1(M, \mathbb{R}) = 0$ and let $\rho\colon \pi_1(M) \to U(n)$ be nontrivial. Then $h^0(M, E_\rho) < n$. The Poincaré duality and Theorem 3.2 give $\chi(M, E) = 2h^0(M, E_\rho) - 2h^1(m, E_\rho)$. Similarly, $\chi(M) = 2 - 2b_1(M)$. Since $b_1(M) = 0$,

we get $\chi(M, E) < n\chi(M)$. On the other hand, $c_i(E_\rho) = 0$ because E_ρ is flat, so by AS index theorem we get $\chi(M, E) = n\chi(M)$. This contradiction shows that $H^1(M, \mathbb{R}) \neq 0$, hence the result. The proof for the self-dual manifolds with positive scalar curvature is parallel, but one looks at the modified complex $\Omega^0(E_\rho) \overset{d}{\longrightarrow} \Omega^1(E_\rho) \overset{\mathcal{D}}{\longrightarrow} \Omega^2_-(E_\rho)$.

3.4 COROLLARY: *Under the conditions of 3.3, let $\pi_1(M)$ be finite. Then either $\pi_1(M) = 1$ or $\pi_1(M) = \mathbb{Z}_2$.*

Proof: Consider the orientable covering of M and apply Theorem 3.3.

The f.p. groups which do not yield the conditions of Theorem 3.3 are rare. They may not admit any nontrivial linear representation over any field and any subgroup of finite index.

4. Concluding remarks

4.1   Here we address the problem, stated in 1.3:

Is any compact simply-connected four-manifold, with curvature, positive on complex isotropic two-planes, diffeomorphic to a four-sphere?

A weaker question is:

4.2   Is any compact four-manifold M, with pointwise $\frac{1}{4}$-pinched curvature, diffeomorphic to S^4?

There are several ways to prove that a δ-pinched manifold, with δ sufficiently close to 1, is a standard sphere [12], [14]. We sketch here an approach to settle 1.3, as follows:

4.3   Fix a $SU(2)$-principal bundle η over M with $(c_2, [M]) = -1$. Consider the moduli space $\mathcal{M}$ of SD-connections in η. Then the following result holds:

LEMMA (4.3): *The metric of M behaves like a generic metric, that is, the canonical compactification $\bar{\mathcal{M}}$ of $\mathcal{M}$ is a smooth compact 5-manifold with M as a boundary.*

Proof: The standard analysis shows ([8], p.69) that for $\mathcal{M}$ to be a smooth manifold it is enough that for any SD-connection ∇, the operator

$$\mathcal{D}: \Omega^1(ad\,\eta) \to \Omega^2_-(ad\,\eta)$$

will be surjective. Since $\mathcal{D}\mathcal{D}^* = \nabla^*\nabla - 2W^- + s/3$ for ∇ self-dual ([8], p.111), and since $s/3 - 2W^-$ is a positive operator by the lemma $\mathcal{M}$ is smooth. Since $H^2(M, \mathbb{R}) = 0$ there are no reducible connections, so $\partial\bar{\mathcal{M}} = M$ by Donaldson's theorem.

4.4   Now, there is a canonical Finsler metric on $\mathcal{M}$, coming from L^4-metric on $\Omega^1(ad\,\eta)$. This metric is invariant under conformal changes of the metric on M. In case $M = (S^4, can)$ this Finsler metric is in fact a (Riemannian) hyperbolic metric of B^5. In any case, by standard methods of Finsler manifolds, one finds a canonical "osculating" Riemannian metric g in $\mathcal{M}$. The direct computation shows the following:

LEMMA (4.4): *If M is δ-pinched with δ close to 1, then the curvature of $(\mathcal{M}, g)$ is pinched negative.*

Then it follows immediately that $M = \partial\bar{\mathcal{M}} \approx S^4$. We hope that a finer analysis will be helpful to settle 1.3 and 4.1.

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